Exercise 3E: Solutions

1 **a** A simple start is often to subtract the equations.

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$
$$\text{If } x = 0,\ y = 0$$
$$\text{If } x = 1,\ y = 1$$

The points of intersection are $(0,0)$ and $(1,1)$.

b Subtract the equations:

$$2x^2 - x = 0$$
$$x(2x-1) = 0$$
$$x = 0 \text{ or } x = \frac{1}{2}$$

If $x = 0,\ y = 0$

If $x = \frac{1}{2},\ y = \frac{1}{2}$

The points of intersection are $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

c Subtract the equations:

$$x^2 - 3x - 1 = 0$$
$$x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times -1}}{2}$$
$$= \frac{3 \pm \sqrt{13}}{2}$$
$$= \frac{3 + \sqrt{13}}{2} \text{ or } \frac{3 - \sqrt{13}}{2}$$
$$\text{If } x = \frac{3+\sqrt{13}}{2},\ y = 2 \times \frac{3+\sqrt{13}}{2} + 1$$
$$= 4 + \sqrt{13}$$
$$\text{If } x = \frac{3-\sqrt{13}}{2},\ y = 2 \times \frac{3-\sqrt{13}}{2} + 1$$
$$= 4 - \sqrt{13}$$

The points of intersection are

$$\left(\frac{3+\sqrt{13}}{2}, 4+\sqrt{13}\right) \text{ and } \left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right).$$

2 **a** Substitute $y = 16 - x$ into $x^2 + y^2 = 178$

$$x^2 + (16-x)^2 = 178$$
$$x^2 + 256 - 32x + x^2 = 178$$
$$2x^2 - 32x + 78 = 0$$
$$x^2 - 16x + 39 = 0$$
$$(x-3)(x-13) = 0$$
$$x = 3 \text{ or } x = 13$$
$$\text{If } x = 3, y = 16 - x = 13$$
$$\text{If } x = 13, y = 16 - x = 3$$

The points of intersection are $(3, 13)$ and $(13, 3)$.

b Substitute $y = 15 - x$ into $x^2 + y^2 = 125$.

$$x^2 + (15-x)^2 = 125$$
$$x^2 + 225 - 30x + x^2 = 125$$

$$2x^2 - 30x + 100 = 0$$
$$x^2 - 15x + 50 = 0$$
$$(x-5)(x-10) = 0$$
$$x = 5 \text{ or } x = 10$$

If $x = 5,\ y = 15 - x = 10$
If $x = 10,\ y = 15 - x = 5$

The points of intersection are $(5, 10)$ and $(10, 5)$.

c Substitute $y = x - 3$ into $x^2 + y^2 = 185$.

$$x^2 + (x-3)^2 = 185$$
$$x^2 + x^2 - 6x + 9 = 185$$
$$2x^2 - 6x - 176 = 0$$
$$x^2 - 3x - 88 = 0$$
$$(x-11)(x+8) = x = 0$$
$$x = 11 \text{ or } x = -8$$

If $x = 11, y = x - 3 = 8$
If $x = -8, y = x - 3 = -11$

The points of intersection are $(11, 8)$ and $(-8, -11)$.

d Substitute $y = 13 - x$ into $x^2 + y^2 = 97$.

$$x^2 + (13-x)^2 = 97$$
$$x^2 + 169 - 26x + x^2 = 97$$
$$2x^2 - 26x + 72 = 0$$
$$x^2 - 13x + 36 = 0$$
$$(x-4)(x-9) = 0$$
$$x = 4 \text{ or } x = 9$$

If $x = 4,\ y = 13 - x = 9$
If $x = 9,\ y = 13 - x = 4$

The points of intersection are $(4, 9)$ and $(9, 4)$.

e Substitute $y = x - 4$ into $x^2 + y^2 = 106$.

$$x^2 + (x-4)^2 = 106$$
$$x^2 + x^2 - 8x + 16 = 106$$
$$2x^2 - 8x - 90 = 0$$
$$x^2 - 4x - 45 = 0$$
$$(x-9)(x+5) = 0$$
$$x = 9 \text{ or } x = -5$$

If $x = 9,\ y = x - 4 = 5$
If $x = -5,\ y = x - 4 = -9$

The points of intersection are $(9, 5)$ and $(-5, -9)$.

3 a Substitute $y = 28 - x$ into $xy = 187$.

$$x(28-x) = 187$$
$$28x - x^2 = 187$$
$$x^2 - 28x + 187 = 0$$
$$(x-11)(x-17) = 0$$
$$x = 11 \text{ or } x = 17$$

If $x = 11,\ y = 28 - x = 17$
If $x = 17,\ y = 28 - x = 11$

The points of intersection are $(11, 17)$ and $(17, 11)$.

b Substitute $y = 51 - x$ into $xy = 518$.

$$\begin{aligned} x(51-x) &= 518 \\ 51x - x^2 &= 518 \\ x^2 - 51x + 518 &= 0 \\ (x-14)(x-37) &= 0 \\ x &= 14 \text{ or } x = 37 \end{aligned}$$

$$\begin{aligned} &\text{If } x = 14,\ y = 51 - x = 37 \\ &\text{If } x = 37,\ y = 51 - x = 14 \end{aligned}$$

The points of intersection are $(14, 37)$ and $(37, 14)$.

c Substitute $y = x - 5$ into $xy = 126$.

$$\begin{aligned} x(x-5) &= 126 \\ x^2 - 5x &= 126 \\ x^2 - 5x - 126 &= 0 \\ (x-14)(x+9) &= 0 \\ x &= 14 \text{ or } x = -9 \end{aligned}$$

$$\text{If } x = 14,\ y = x - 5 = 9$$

$$\text{If } x = -9,\ y = x - 5 = -14$$

The points of intersection are $(14, 9)$ and $(-9, -14)$.

4 Substitute $y = 2x$ into the equation of the circle.

$$\begin{aligned} (x-5)^2 + (2x)^2 &= 25 \\ x^2 - 10x + 25 + 4x^2 &= 25 \\ 5x^2 - 10x &= 0 \\ 5x(x-2) &= 0 \\ x &= 0 \text{ or } x = 2 \end{aligned}$$

$$\text{If } x = 0, y = 2x = 0$$

$$\text{If } x = 2, y = 2x = 4$$

The points of intersection are $(0, 0)$ and $(2, 4)$.

5 Substitute $y = x$ into the equation of the second curve.

$$\begin{aligned} x &= \frac{1}{x-2} + 3 \\ x(x-2) &= 1 + 3(x-2) \\ x^2 - 2x &= 1 + 3x - 6 \\ x^2 - 5x + 5 &= 0 \\ x &= \frac{5 \pm \sqrt{25 - 4 \times 1 \times 5}}{2} \\ &= \frac{5 \pm \sqrt{5}}{2} \\ &= \frac{5 + \sqrt{5}}{2} \text{ or } \frac{5 - \sqrt{5}}{2} \end{aligned}$$

Since $y = x$, the points of intersection are

$\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$ and $\left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$.

6 Substitute $x = 3y$ into the equation of the circle.

$$\begin{aligned} 9y^2 + y^2 - 30y - 5y + 25 &= 0 \\ 10y^2 - 35y + 25 &= 0 \\ 2y^2 - 7y + 5 &= 0 \\ (2y-5)(y-1) &= 0 \end{aligned}$$

$$y = \frac{5}{2} \text{ or } y = 1$$

If $y = \frac{5}{2},\ x = 3y = \frac{15}{2}$

If $y = 1,\ x = 3y = 3$

The points of intersection are $\left(\frac{15}{2}, \frac{5}{2}\right)$ and $(3, 1)$.

7 Make y the subject in $\frac{y}{4} - \frac{x}{5} = 1$.

$$\frac{y}{4} = \frac{x}{5} + 1$$

$$y = \frac{4x}{5} + 4$$

Substitute into $x^2 + 4x + y^2 = 12$.

$$x^2 + 4x + \left(\frac{4x}{5} + 4\right)^2 = 12$$

$$x^2 + 4x + \frac{16x^2}{25} + \frac{32x}{5} + 16 = 12$$

$$25x^2 + 100x + 16x^2 + 160x + 400 = 300$$

$$41x^2 + 260x + 100 = 0$$

$$x = \frac{-260 \pm \sqrt{67\,600 - 4 \times 41 \times 100}}{82}$$

$$= \frac{-260 \pm \sqrt{51\,200}}{82}$$

$$= \frac{-260 \pm \sqrt{25\,600 \times 2}}{82}$$

$$= \frac{-260 \pm 160\sqrt{2}}{82}$$

$$= \frac{-130 \pm 80\sqrt{2}}{41}$$

If $x = \frac{-130 + 80\sqrt{2}}{41}$,

$$y = \frac{4 \times (-130 + 80\sqrt{2})}{5 \times 41} + 4$$

$$= \frac{4 \times (-26 + 16\sqrt{2})}{41} + \frac{4 \times 41}{41}$$

$$= \frac{-104 + 64\sqrt{2} + 164}{41}$$

$$= \frac{60 + 64\sqrt{2}}{41}$$

Likewise, if $x = \frac{-130 - 80\sqrt{2}}{41}$,

$$y = \frac{60 - 64\sqrt{2}}{41}$$

The points of intersection are

$\left(\frac{-130 + 80\sqrt{2}}{41}, \frac{60 + 64\sqrt{2}}{41}\right)$ and $\left(\frac{-130 - 80\sqrt{2}}{41}, \frac{60 - 64\sqrt{2}}{41}\right)$.

8 Subtract the second equation from the first.

$$\frac{1}{x+2} - 3 + x = 0$$

$$1 - 3(x+2) + x(x+2) = 0$$

$$1 - 3x - 6 + x^2 + 2x = 0$$

$$x^2 - x - 5 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times -5}}{2}$$

$$= \frac{1 \pm \sqrt{21}}{2}$$

If $x = \frac{1+\sqrt{21}}{2}$, $y = -x = \frac{-1-\sqrt{21}}{2}$

If $x = \frac{1-\sqrt{21}}{2}$, $y = -x = \frac{-1+\sqrt{21}}{2}$

The points of intersection are

$\left(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right)$ and $\left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

9 Substitute $y = \frac{9x+4}{4}$ into the equation of the parabola.

$$\left(\frac{9x+4}{4}\right)^2 = 9x$$

$$\frac{(9x+4)^2}{16} = 9x$$

$$(9x+4)^2 = 9x \times 16$$

$$81x^2 + 72x + 16 = 144x$$

$$81x^2 - 72x + 16 = 0$$

$$(9x-4)^2 = 0$$

$$x = \frac{4}{9}$$

$$y = \frac{9x+4}{4}$$

$$= \frac{4+4}{4} = 2 \left(\frac{4}{9}, 2\right)$$

Note: Substitute into the linear equation, as substituting into the quadratic introduces a second answer that is not actually a solution.

10 Substitute $y = 2x + 3\sqrt{5}$ into the equation of the circle.

$$x^2 + (2x + 3\sqrt{5})^2 = 9$$

$$x^2 + 4x^2 + 12\sqrt{5x} + 45 = 9$$

$$5x^2 + 12\sqrt{5x} + 36 = 0$$

$$x^2 + \frac{12\sqrt{5}}{5}x + \frac{36}{5} = 0$$

$$x^2 + \frac{2 \times 6\sqrt{5}}{5}x + \frac{(6\sqrt{5})^2}{25} = 0$$

$$\left(x + \frac{6\sqrt{5}}{5}\right)^2 = 0$$

$$x = -\frac{6\sqrt{5}}{5}$$

$$y = 2x + 3\sqrt{5}$$

$$= -\frac{12\sqrt{5}}{5} + \frac{15\sqrt{5}}{5}$$

$$= \frac{3\sqrt{5}}{5} \left(-\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$$

11 Substitute $y = \frac{1}{4}x + 1$ into $y = -\frac{1}{x}$.

$$\begin{aligned} \frac{1}{4}x + 1 &= -\frac{1}{x} \\ \frac{x+4}{4} &= -\frac{1}{x} \\ x(x+4) &= -4 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \\ x &= -2 \\ y &= -\frac{1}{x} \\ &= \frac{1}{2} \left(-2, \frac{1}{2}\right) \end{aligned}$$

12 Substitute $y = x - 1$ into $y = \frac{2}{x-2}$.

$$\begin{aligned} x - 1 &= \frac{2}{x-2} \\ (x-1)(x-2) &= 2 \\ x^2 - 3x + 2 &= 2 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

If $x = 0,\ y = x - 1 = -1$

If $x = 3,\ y = x - 1 = 2$

The points of intersection are $(0,\ -1)$ and $(3, 2)$.

13a

$$\begin{aligned} 2x^2 - 4x + 1 &= 2x^2 - x - 1 \\ -3x &= -2 \\ x &= \frac{2}{3} \\ y &= -\frac{7}{9} \end{aligned}$$

b

$$\begin{aligned} -2x^2 + x + 1 &= 2x^2 - x - 1 \\ 4x^2 - 2x - 2 &= 0 \\ 2x^2 - x - 1 &= 0 \\ (2x+1)(x-1) &= 0 \\ x = -\frac{1}{2} &\text{ or } x = 1 \end{aligned}$$

Solutions: $\left(\frac{-1}{2}, 0\right), (1, 0)$

c

$$\begin{aligned} x^2 + x + 1 &= x^2 - x - 2 \\ 2x + 3 &= 0 \\ x &= -\frac{3}{2} \\ y &= \frac{7}{4} \end{aligned}$$

d

$$\begin{aligned} 3x^2 + x + 2 &= x^2 - x + 2 \\ 2x^2 + 2x- &= 0 \\ 2x(x+1) &= 0 \\ x = 0 &\text{ or } x = -1 \end{aligned}$$

Solutions: $(-1, 4), (0, 2)$

14a $k = -2$, $k = 1$

b $-10 < c < 10$

c $p = 5$